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# The Sum of Like Powers of the Zeros of the Riemann Zeta Function

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**Abstract.** In this paper we discuss a method of evaluating the sum $\sigma_r = \sum \rho^{-r}$ where $r$ is an integer greater than 1 and the sum is taken over all the complex zeros of $\zeta(s)$, the Riemann zeta function. The method requires the coefficients of the Maclaurin expansion of the entire function $f(s) = (s-1)\zeta(s)$. These are obtained from a limit theorem of Sitaramachandrarao by the use of the Euler-Maclaurin summation formula. The sum $\sigma_r$ is then obtained from the logarithmic derivative of the function $f(s)$. A table of $\sigma_r$ is given to 30 decimals for $r = 2(1)26$.

Despite the vast literature and machine computing on the zeros of

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

no one has attempted to evaluate

$$\sigma_2 = \sum_{\rho} \rho^{-2}, \tag{1}$$

the sum extending over all the complex zeros $\rho$ of $\zeta(s)$.

The direct approach, summing over the known zeros, is not effective. For example, we find the first 50 zeros sum to

$$\sum_{|\rho| \le 88.8} \rho^{-2} = -.034721,$$

whereas the infinite series (1) is nearly a time and a half as much, as we shall see. The Dirichlet series representation of functions of a complex variable is not conducive to the solution of this problem, which needs a power series approach.

In a letter to Hermite [1], Stieltjes gave the expression

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n, \tag{2}$$

where $\gamma_n$ is a generalization of Euler's constant $\gamma$,

$$\gamma_n = \lim_{N \to \infty} \left\{ \sum_{k=1}^{N} \frac{(\log k)^n}{k} - \frac{(\log N)^{n+1}}{n+1} \right\}.$$

Briggs and Chowla [2] give two proofs of this result. Liang and Todd [3] call them Stieltjes constants and give a table of $\gamma_n$ for $n \le 19$ to 15 significant decimals, improving the earlier results by Jensen [4] and Gram [5].

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The power series in (2) is an entire function of $s$ and hence has an infinite radius of convergence.

To find, more generally,

$$\sigma_k = \sum_\rho \rho^{-k} \tag{3}$$

we need an expansion in powers of $s$, not $(s-1)$. In executing the transformation so much accuracy is lost that it was decided to abandon the method. What we need is a method of computing $\varsigma^{(k)}(0)$. It is well known that

$$\varsigma(0) = -\frac{1}{2}, \qquad \varsigma'(0) = -\frac{1}{2}\log 2\pi.$$

Ramanujan [6, p. 25] gave

$$\varsigma''(0) = -\frac{1}{2}(\log 2\pi)^2 + \frac{\pi^2}{24} - a_1,$$

and Apostol [7] gives a formula for $\varsigma^{(k)}(0)$ and a table for $k \leq 18$ to 15D in terms of $a_n$ defined by

$$\sum_{n=0}^{\infty} a_n(s-1)^n = \Gamma(s)\varsigma(s) + \frac{1}{1-s}.$$

We must regard the $a$'s as unknowns and as difficult to approximate as $\varsigma^{(k)}(0)$ itself.

It was decided to use an *ab initio* approach *via* a recent theorem of Sitaramachandrarao [8] to the effect that

$$\varsigma(s) + \frac{1}{1-s} = \sum_{n=0}^{\infty} \frac{(-1)^n \delta_n}{n!} s^n, \tag{4}$$

where

$$\delta_n = \lim_{N\to\infty} \left\{ \sum_{k=1}^{N} (\log k)^n - \int_1^N (\log t)^n \, dt - \frac{1}{2}(\log N)^n \right\}. \tag{5}$$

This is an analogue of Stieltjes' theorem.

To calculate $\delta_n$ we use the Euler-Maclaurin summation formula [9, p. 806]

$$\begin{aligned} \sum_{i=0}^{m} F(a+i) &= \int_a^b F(t)\,dt + \frac{1}{2}F(b) + \frac{1}{2}F(a) \\ &\quad + \sum_{k=1}^{p-1} \frac{B_{2k}}{(2k)!}\{F^{(2k-1)}(b) - F^{(2k-1)}(a)\} + R_p \end{aligned} \tag{6}$$

where $m = b - a$ and $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ are the Bernoulli numbers. We also use the Stirling numbers of the first kind. These latter are denoted by $s(i,j)$ and are generated by [9, p. 824]

$$\sum_{j=0}^{i} s(i,j)x^j = x(x-1)(x-2)\cdots(x-i+1)$$

and enjoy the recurrence

$$s(i+1,j) = s(i,j-1) - is(i,j) \qquad (s(0,0) = 1). \tag{7}$$

We have the following lemma:

LEMMA. *If $h$ is a nonnegative integer, then*

$$\frac{d^h}{dx^h}\{(\log x)^n\} = x^{-h} n! \sum_{k=0}^{n} \frac{(\log x)^k}{k!} s(h, n-k).$$

*Proof.* The proof is by induction on $h$. The lemma is easily seen to be true for $h = 0$, since $s(0, j) = 0$, except for $s(0,0) = 1$. If true for $h$, we have

$$\begin{aligned}\frac{d^{h+1}}{dx^{h+1}}\{(\log x)^n\} &= -hx^{-(h+1)} n! \sum_{k=0}^{n} \frac{(\log x)^k}{k!} s(h, n-k) \\ &\quad + x^{-(h+1)} n! \sum_{k=1}^{n} \frac{(\log x)^{k-1}}{(k-1)!} s(h, n-k) \\ &= x^{-(h+1)} n! \sum_{k=0}^{n} \frac{(\log x)^k}{k!} (s(h, n-k-1) - hs(h, n-k)) \\ &= x^{-(h+1)} n! \sum_{k=0}^{n} \frac{(\log k)^k}{k!} s(h+1, n-k).\end{aligned}$$

Thus the induction is complete.

Before we apply (6) we fix a number $\nu \leq N$ and write

$$\sum_{k=1}^{N} (\log k)^n = \sum_{k=1}^{\nu-1} (\log k)^n + \sum_{k=\nu}^{N} (\log k)^n = S_1 + S_2.$$

We plan to evaluate $S_1$ on our machine. As to $S_2$, we apply (6) with $m = N - \nu$, $b = N$, $a = \nu$ and $F(t) = (\log t)^n$ to get from the lemma

$$\begin{aligned}S_2 = &\int_{\nu}^{N} (\log t)^n dt + \frac{1}{2}(\log N)^n + \frac{1}{2}(\log \nu)^n \\ &+ \sum_{k=1}^{p-1} \frac{B_{2k}}{(2k)!} \sum_{j=0}^{n-1} \frac{n!}{j!} s(2k-1, n-j)(\log N)^j N^{1-2k} \\ &- \sum_{k=1}^{p-1} \frac{B_{2k}}{(2k)!} \sum_{j=0}^{n-1} \frac{n!}{j!} s(2k-1, n-j)(\log \nu)^j \nu^{1-2k} + R_p.\end{aligned}$$

Substituting this into (5), cancelling and letting $N \to \infty$ we find

$$\begin{aligned}\delta_n = &\sum_{k=1}^{\nu} (\log k)^n - \int_{1}^{\nu} (\log t)^n \, dt - \frac{1}{2}(\log \nu)^n \\ &- \sum_{k=1}^{p-1} \frac{B_{2k}}{(2k)!\nu^{2k-1}} \sum_{j=0}^{n-1} \frac{n!}{j!} s(2k-1, n-j)(\log \nu)^j + R_p.\end{aligned} \tag{8}$$

TABLE 1

| $n$ | $\delta_n$ |
|---|---|
| 0 | .5000 0000 0000 0000 0000 0000 0000 00 |
| 1 | − .0810 6146 6795 3272 5821 9670 2635 94 |
| 2 | − .0063 5645 5908 5848 5121 0100 0267 30 |
| 3 | .0047 1116 6862 2544 4776 1060 8133 66 |
| 4 | .0028 9681 1986 2920 4101 2780 4722 59 |
| 5 | .0002 3290 7558 4547 2453 5985 8377 96 |
| 6 | − .0009 3682 5130 0509 2950 4283 5085 45 |
| 7 | − .0008 4982 3765 0016 6915 1706 0276 02 |
| 8 | − .0002 3243 1735 5115 5958 2855 6900 64 |
| 9 | .0003 3058 9663 6122 9644 5256 1272 50 |
| 10 | .0005 4323 4115 7797 0847 2231 9889 43 |
| 11 | .0003 7549 3172 9072 6365 0467 0308 84 |
| 12 | − .0000 1960 3536 2810 1391 9766 4840 25 |
| 13 | − .0004 0724 1232 5630 3314 3432 1213 67 |
| 14 | − .0005 7049 2013 2817 7771 5641 2913 84 |
| 15 | − .0003 9392 7078 9812 0442 1827 6608 19 |
| 16 | .0000 8345 8805 8255 0168 1727 6488 05 |
| 17 | .0006 6094 3729 6285 9689 6169 4029 98 |
| 18 | .0010 2622 7286 5408 5400 2177 0141 55 |
| 19 | .0008 6557 5776 7792 8299 1576 0724 14 |
| 20 | .0000 1929 3671 7837 0514 0106 3299 76 |
| 21 | − .0013 5690 6052 1345 4946 1149 1378 33 |
| 22 | − .0026 9215 6458 7532 9128 4034 2571 09 |
| 23 | − .0030 5138 5621 2416 2713 8845 4373 86 |
| 24 | − .0014 2429 1849 4185 4585 3222 1867 92 |
| 25 | .0027 0778 9212 8860 0678 8197 4821 92 |
| 26 | .0086 0288 0969 2793 2425 6140 4520 25 |
| 27 | .0135 5616 2030 9835 3962 1697 3716 23 |
| 28 | .0127 7851 3326 6914 1273 7021 7809 89 |
| 29 | .0005 7602 6175 9930 1208 9937 4469 58 |
| 30 | − .0264 6570 4147 0797 5269 3730 4048 60 |

This expression for $\delta_n$ depends on the two parameters $\nu$ and $p$. The optimal choice of the parameters depends on $n$ but this dependence complicates the programming. There is also a dependence on the desired accuracy of $\delta_n$ and accordingly we must use sufficiently precise logarithms. This is no simple matter. In experimenting with (8), $|\delta_n|$ grows very slowly, but the four terms of (8) are decidedly unbounded. In particular, therefore, the sum of the first three terms is nearly the negative of the fourth term. For example if $\nu = 180$, $p = 15$ and $n = 24$, the first three terms contribute

$$316\ 4420\ 8836\ 5362.2542\ 0772\ldots$$

whereas the fourth term is

$$-316\ 4420\ 8836\ 5362.2556\ 3201\ldots.$$

This gives

$$\delta_{24} = -.0014\ 2429.$$

Only six significant decimals remain from the destructive cancellation of numbers of 23 significant decimals. This loss of significance increases with $n$, but is relatively harmless for $n \leq 6$.

TABLE 2

| $n$ | $c_n$ |
|---|---|
| 0 | .5000 0000 0000 0000 0000 0000 0000 00 |
| 1 | .0810 6146 6795 3272 5821 9670 2635 94 |
| 2 | − .0031 7822 7954 2924 2560 5050 0133 65 |
| 3 | − .0007 8519 4477 0424 0796 0176 8022 28 |
| 4 | .0001 2070 0499 4288 3504 2199 1863 44 |
| 5 | − .0000 0194 0896 3204 5603 7799 8819 82 |
| 6 | − .0000 0130 1146 0139 5962 4311 5048 73 |
| 7 | .0000 0016 8615 8263 8922 0069 7829 42 |
| 8 | − .0000 0000 5764 6759 7994 9394 4160 64 |
| 9 | − .0000 0000 0911 0164 8923 1416 5709 22 |
| 10 | .0000 0000 0149 7007 5941 9011 3735 21 |
| 11 | − .0000 0000 0009 4068 9566 5666 1769 10 |
| 12 | − .0000 0000 0000 0409 2582 6304 1583 15 |
| 13 | .0000 0000 0000 0653 9904 8058 9101 52 |
| 14 | − .0000 0000 0000 0065 4396 8749 8919 19 |
| 15 | .0000 0000 0000 0003 0124 2487 1366 79 |
| 16 | .0000 0000 0000 0000 0398 8894 7063 11 |
| 17 | − .0000 0000 0000 0000 0185 8215 0433 79 |
| 18 | .0000 0000 0000 0000 0016 0288 5638 53 |
| 19 | − .0000 0000 0000 0000 0000 7115 5827 39 |
| 20 | .0000 0000 0000 0000 0000 0007 9303 12 |
| 21 | .0000 0000 0000 0000 0000 0026 5586 42 |
| 22 | − .0000 0000 0000 0000 0000 0002 3951 55 |
| 23 | .0000 0000 0000 0000 0000 0000 1180 33 |
| 24 | − .0000 0000 0000 0000 0000 0000 0022 96 |
| 25 | − .0000 0000 0000 0000 0000 0000 0001 75 |
| 26 | .0000 0000 0000 0000 0000 0000 0000 21 |
| 27 | − .0000 0000 0000 0000 0000 0000 0000 01 |

It was decided to run a program for $\delta_n$ that exploited certain recursive features of (8). For example, if

$$I_n = \int_1^{\nu} (\log t)^n \, dt$$

then

$$I_n = \nu(\log \nu)^n - nI_{n-1} \qquad (I_0 = \nu - 1).$$

It was also decided to compute $\delta_1, \delta_2, \ldots, \delta_{30}$ with accuracy sufficient to determine $\delta_n$ from 44 significant figures for $n = 1$ to about 10 for $n = 30$. The details of this program with its 1024 instructions and its multiprecision subroutines will not be given. The estimation of the remainder $R_p$ was avoided by the following considerations. The denominator factor $\nu^{2k-1}$ assures us of terms in the asymptotic series that decrease by a factor of at least $10^4$ for $\nu > 100$. The machine was instructed to choose $p$ so that the last term it computed was less than $10^{-40}$, so that $|R_p| < 10^{-40}$. Two runs were made with $\nu = 180$ and $\nu = 200$. The two results were compared and only the digits that were common to the two runs were retained. Table 1 gives the values of $\delta_n$.

TABLE 3

| $n$ | $d_n$ |
|---|---|
| 0 | .5000 0000 0000 0000 0000 0000 0000 00 |
| 1 | .4189 3853 3204 6727 4178 0329 7364 06 |
| 2 | .0842 3969 4749 6196 8382 4720 2769 59 |
| 3 | −.0023 9303 3477 2500 1764 4873 2111 37 |
| 4 | −.0009 0589 4976 4712 4300 2375 9885 72 |
| 5 | .0001 2264 1395 7492 9107 9999 0683 26 |
| 6 | −.0000 0063 9750 3064 9641 3488 3771 09 |
| 7 | −.0000 0146 9761 8403 4884 4381 2878 15 |
| 8 | .0000 0017 4380 5023 6916 9464 1990 06 |
| 9 | −.0000 0000 4853 6594 9071 7977 8451 42 |
| 10 | −.0000 0000 1060 7172 4865 0427 9444 43 |
| 11 | .0000 0000 0159 1076 5508 4677 5504 30 |
| 12 | −.0000 0000 0009 3659 6983 9362 0185 94 |
| 13 | −.0000 0000 0000 1063 2487 4363 0684 68 |
| 14 | .0000 0000 0000 0719 4301 6808 8020 72 |
| 15 | −.0000 0000 0000 0068 4521 1237 0285 98 |
| 16 | .0000 0000 0000 0002 9725 3592 4303 68 |
| 17 | .0000 0000 0000 0000 0584 7109 7496 90 |
| 18 | −.0000 0000 0000 0000 0201 8503 6072 31 |
| 19 | .0000 0000 0000 0000 0016 7404 1465 92 |
| 20 | −.0000 0000 0000 0000 0000 7123 5130 51 |
| 21 | −.0000 0000 0000 0000 0000 0018 6283 30 |
| 22 | .0000 0000 0000 0000 0000 0028 9537 98 |
| 23 | −.0000 0000 0000 0000 0000 0002 5131 88 |
| 24 | .0000 0000 0000 0000 0000 0000 1203 28 |
| 25 | −.0000 0000 0000 0000 0000 0000 0021 21 |
| 26 | −.0000 0000 0000 0000 0000 0000 0001 96 |
| 27 | .0000 0000 0000 0000 0000 0000 0000 23 |
| 28 | −.0000 0000 0000 0000 0000 0000 0000 01 |

We use the following notation.

$$\varsigma(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}(-1)^n \delta_n s^n/n! = \frac{1}{s-1} + \sum_{n=0}^{\infty} c_n s^n,$$

$$(9)\qquad \begin{gathered} (s-1)\varsigma(s) = \sum_{n=0}^{\infty} d_n s^n \qquad (d_0 = \tfrac{1}{2},\ d_n = c_{n-1} - c_n), \\ g(s) = 2(s-1)\varsigma(s), \\ \frac{g'(s)}{g(s)} = \sum_{n=0}^{\infty} b_n s^n \qquad \left(b_i = 2\left[(i+1)d_{i+1} - \sum_{j=0}^{i-1} b_j d_{i-j}\right],\ i = 0, 1, \ldots\right). \end{gathered}$$

The coefficients $c_n$, $d_n$ and $b_n$ are given in Tables 2, 3 and 4.

In terms of the complex zeros of $\varsigma(s)$, we have the Weierstrass expansion [9, p. 807]

$$\varsigma(s) = \frac{\exp\{s(\log 2\pi - 1 - \frac{1}{2}\gamma)\}}{2(s-1)\Gamma(\frac{1}{2}s+1)} \prod_{\rho}\left(1 - \frac{s}{\rho}\right) e^{s/\rho}.$$

Multiplying this by $2(s-1)$ and taking logarithms, we get

$$(10)\qquad \log g(s) = s(\log 2\pi - 1 - \tfrac{1}{2}\gamma) - \log\Gamma(\tfrac{1}{2}s+1) + \sum_{\rho}\left(\log\left(1 - \frac{2}{\rho}\right) + \frac{s}{\rho}\right).$$

TABLE 4

| $n$ | $b_n$ |
|---|---|
| 0 | .8378 7706 6409 3454 8356 0659 4728 11 |
| 1 | − .3650 7919 9416 2520 0636 0995 8012 82 |
| 2 | .1503 6827 1126 4013 9159 7729 9384 28 |
| 3 | − .0677 1882 9328 2078 2648 8077 0023 41 |
| 4 | .0324 0327 7254 8745 4759 0119 3703 58 |
| 5 | − .0158 9570 3907 0900 7904 1777 7359 37 |
| 6 | .0078 7773 3303 7374 2178 9458 0295 21 |
| 7 | − .0039 2217 8441 5163 3102 6458 7082 25 |
| 8 | .0019 5704 7613 9640 7027 4475 9977 97 |
| 9 | − .0009 7753 3758 7755 4500 1810 1820 06 |
| 10 | .0004 8852 2553 1984 2408 3329 9750 07 |
| 11 | − .0002 4420 0704 7536 6687 4030 0855 75 |
| 12 | .0001 2208 5292 1557 0353 0612 2853 56 |
| 13 | − .0000 6103 8894 5393 5540 2858 5591 25 |
| 14 | .0000 3051 8511 6038 9679 6797 0572 13 |
| 15 | − .0000 1525 9022 2512 7337 4837 3822 39 |
| 16 | .0000 0762 9452 7984 4386 1451 9963 50 |
| 17 | − .0000 0381 4711 8274 4317 6944 9679 33 |
| 18 | .0000 0190 7352 2724 3941 6206 2375 85 |
| 19 | − .0000 0095 3675 2261 7534 0545 9787 34 |
| 20 | .0000 0047 6837 3856 2249 5072 2941 87 |
| 21 | − .0000 0023 8418 6359 5259 2546 3299 24 |
| 22 | .0000 0011 9209 3037 6290 3924 6264 77 |
| 23 | − .0000 0005 9604 6483 2831 5559 1049 72 |
| 24 | .0000 0002 9802 3232 7590 8932 5264 51 |
| 25 | − .0000 0001 4901 1614 1589 8126 7864 25 |
| 26 | .0000 0000 7450 5806 5243 5956 8196 20 |
| 27 | − .0000 0000 3725 2903 1233 9864 7635 75 |
| 28 | .0000 0000 1862 6451 5270 0431 1299 19 |
| 29 | − .0000 0000 0931 3225 7548 2844 7777 96 |
| 30 | .0000 0000 0465 6612 8752 4580 4463 06 |

We note that

$$\sum_\rho \left(\log\left(1-\frac{s}{\rho}\right)+\frac{s}{\rho}\right) = -\sum_\rho \sum_{r=2}^{\infty} \frac{s^r}{r\rho^r}.$$

Differentiating (10) with respect to $s$ and using the formula [9, p. 259, 6.3.14],

$$\frac{d}{ds}\log\Gamma\left(\frac{s}{2}+1\right) = \frac{1}{2}\psi\left(\frac{s}{2}+1\right) = -\frac{\gamma}{2} + \sum_{n=2}^{\infty}(-1)^n \frac{\varsigma(n)}{2^n} s^{n-1},$$

we get

$$\frac{g'(s)}{g(s)} = \log 2\pi - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n \varsigma(n)}{2^n} s^{n-1} - \sum_\rho \sum_{r=2}^{\infty} s^{r-1}\rho^{-r}. \tag{11}$$

Identifying the coefficient of $s^m$ in both (11) and (9), we get

$$b_0 = \log 2\pi - 1, \quad b_m = \frac{(-1)^m \varsigma(m+1)}{2^{m+1}} - \sigma_{m+1} \qquad (m > 0),$$

where $\sigma_k$ is defined in (3). Hence,

$$\sigma_r = -\left(\frac{(-1)^r \varsigma(r)}{2^r} + b_{r-1}\right), \qquad r > 1. \tag{12}$$

TABLE 5

| $n$ | $\sigma_n$ |
|---|---|
| 2 | − .0461 5431 7295 8046 0275 7107 9903 79 |
| 3 | − .0001 1115 8231 4521 0592 2762 6682 39 |
| 4 | .0000 7362 7221 2616 8951 8326 7713 07 |
| 5 | .0000 0071 5093 3557 6260 7735 8010 94 |
| 6 | − .0000 0028 1436 4169 3876 6261 6067 16 |
| 7 | − .0000 0000 4574 1911 4970 4772 1111 63 |
| 8 | .0000 0000 1268 8681 1095 0760 7190 13 |
| 9 | .0000 0000 0028 2743 7155 0558 8708 93 |
| 10 | − .0000 0000 0005 9977 1484 7151 8745 95 |
| 11 | − .0000 0000 0000 1690 6887 6025 9211 58 |
| 12 | .0000 0000 0000 0287 3815 1300 9259 08 |
| 13 | .0000 0000 0000 0009 8329 6676 1364 99 |
| 14 | − .0000 0000 0000 0001 3792 6293 6173 00 |
| 15 | − .0000 0000 0000 0000 0559 1337 6592 72 |
| 16 | .0000 0000 0000 0000 0065 9812 4839 76 |
| 17 | .0000 0000 0000 0000 0003 1215 8676 76 |
| 18 | − .0000 0000 0000 0000 0000 3138 6496 84 |
| 19 | − .0000 0000 0000 0000 0000 0171 6555 89 |
| 20 | .0000 0000 0000 0000 0000 0014 8239 93 |
| 21 | .0000 0000 0000 0000 0000 0000 9320 09 |
| 22 | − .0000 0000 0000 0000 0000 0000 0694 29 |
| 23 | − .0000 0000 0000 0000 0000 0000 0050 06 |
| 24 | .0000 0000 0000 0000 0000 0000 0003 22 |
| 25 | .0000 0000 0000 0000 0000 0000 0000 27 |
| 26 | − .0000 0000 0000 0000 0000 0000 0000 01 |

In particular,

$$\sigma_2 = -\left(\frac{\pi^2}{24} + b_1\right) = -.0461\ 5431\ 7295\ 8046\ 0275\ 7107\ 9903\ 79$$

and

$$\sigma_3 = -\left(-\frac{\zeta(3)}{8} + b_2\right) = -.0001\ 1115\ 8231\ 4521\ 0592\ 2762\ 6682\ 39.$$

These two numbers are not connected in any obvious way with any other known constants. Their continued fractions show no radical departure from the norm.

Formula (12) may be used to compute other values of $\sigma_r$. The term $\zeta(r)/2^r$ represents the sum of the negative $r$th powers of the trivial zeros $-2n$ $(n = 1, 2, 3, \dots)$ of $\zeta(s)$. Values of $\zeta(n)$ were taken from the 50D tables of Lienard [10]. The values of $\sigma_n$ are given in Table 5.

Table 5 can be used to evaluate such sums as

$$\sum_\rho \frac{1}{\rho(\rho-1)} = \sum_{r=2}^{\infty} \sigma_r = -.0461\ 9141\ 7932\ 2420\ 6762\ 8620\ 4958\ 13$$

and

$$\sum_\rho \frac{1}{\rho(\rho+1)} = \sum_{r=2}^{\infty} (-1)^r \sigma_r = -.0459\ 7052\ 2563\ 8796\ 4241\ 0855\ 6713\ 56.$$

These sums are special cases of

$$\sum_\rho \frac{1}{\rho(\rho-a)} = \sum_{r=2}^{\infty} \sigma_r a^{r-2}.$$

The author owes a debt of gratitude to the referee who went to the trouble of reproducing and extending the values given in Tables 1–5. Not only did he locate several errors, but he made a number of good suggestions which improved the presentation of the method. Also, I have taken the liberty of borrowing some of his values to fill out Table 1.

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